Lecture 21 (2/28/22).

Approximation results. Recall:

Weierstrass Approximation Thm.
Let $I \subseteq \mathbb{R}$ be a compact interval. Then any cont. function f on I can be unif. approximated by polynomials $p(x)$:
$\forall \varepsilon > 0\ \exists p(x)$ s.t. $\sup_I |f-p| < \varepsilon$.

Results like this are fundamental in analysis.

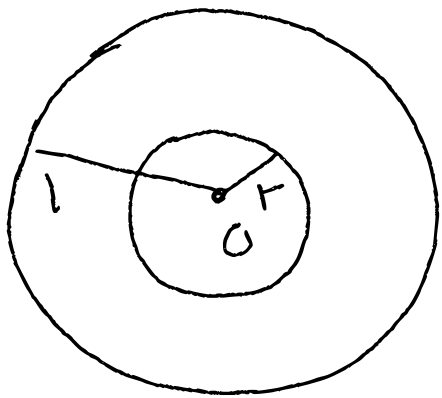
Examples from cplx analysis:

Ex1. Let $f \in H(G)$ and $K = \overline{B(a,r)} \subset\subset G$. Then, f can be unif. approximated by polynomials $p(z)$. Simply take

$$p_k(z) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(z-a)^n \quad \text{(Taylor poly).}$$

Then $p_k \to f$ unif. on K.

Ex2. Polynomials do NOT approximate analytic function on more general compacts. Suppose $G = B(0,R) \setminus \{0\}$ and consider $K = \overline{A(0,r,1)}$, where $0<r<1<R$.



Consider $f(z) = \frac{1}{z} \in H(G)$. This cannot be approximated by poly's $p(z)$.

Why? Max. Principle. Suppose $p(z)$ is a polynomial s.t. $\sup_{|z|=1} |f-p| < \varepsilon$.

Then, since $|f(z)| = 1$ on $|z|=1$, $\Rightarrow$ $|p(z)| \le 1+\varepsilon$ on $|z|=1$. But since p is analytic in $\mathbb{C} \supseteq \overline{B(0,1)}$ $\Rightarrow$ Max. Princ. $|p(z)| \le 1+\varepsilon$ in $\overline{B(0,1)}$. Now, $|f(z)| = \frac{1}{r}$ on $|z|=r$ $\Rightarrow$

$$|f(z)-p(z)| \ge \frac{1}{r} - (1+\varepsilon) = \frac{1}{r} - 1 - \varepsilon$$

on $|z|=r$. If, e.g., $r \le \frac{1}{2}$ then

$|f-p| \ge 1-\varepsilon$. Thus, poly's cannot approximate $f(z) = 1/z$ on K.

On the other hand, by Laurent Series Expansion, any $f \in H(G)$, satisfies

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{1}{z^n} + \sum_{n=0}^{\infty} b_n z^n$$

with unif. conv. on K. Thus, f can be approximated by the rational functions

$$R_k(z) = \sum_{n=1}^{k} a_n \frac{1}{z^n} + \sum_{n=0}^{k} b_n z^n,$$

which have poles at $z=0$ and $z=\infty$.

Let's review the pf of the LSE Thm.

①

Take two circles $\gamma_1 = r_1 e^{it}$, $\gamma_2 = r_2 e^{it}$ as in pic. Then, by Cauchy's Integral Formula, $f \in H(G)$ and $z \in K$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$= f_1(z) + f_2(z).$$

② f_j is analytic in $\mathbb{C} \setminus \{\gamma_j\}$. By expanding $\frac{1}{z-3}$ as a geometric series in $\frac{r_1}{z}$ in the integral for f_1, converging unif. in $|z|>r_1$ and in $\frac{z}{r_2}$ in the integral for f_2, conv. unif. in $|z|<r_2$, we achieve the desired LSE.

With this approach we can obtain approximation results on more general regions / compacts.

<u>Ex3.</u> Let G and K be as in pic:

Here, $a, b \notin G$ and we shall assume there are r_1, r_2, r_3 s.t. $\gamma_1 = a + r e^{-it}$, $\gamma_2 = b + r_2 e^{-it}$, $\gamma_3 = r_3 e^{it}$ are curves in $G \setminus K$ (as is possible in pic) s.t. $f \in H(G) \Rightarrow$ CIF:

$$f(z) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta \quad , \quad z \in K.$$

Expanding $\frac{1}{\zeta - z}$ as geometric series

$$\frac{r_1}{z-a}, \quad \frac{r_2}{z-b}, \quad \frac{z}{r_3} \quad \text{in the}$$

respective integrals gives a Laurent Series type expansion

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{1}{(z-a)^n} + \sum_{n=1}^{\infty} b_n \frac{1}{(z-b)^n} + \sum_{n=0}^{\infty} c_n z^n$$

and hence we conclude that $f \in H(G)$ can be approximated by rational functions w/ poles at $z \in \{a, b, \infty\}$.

This idea can be generalized to produce a general approximation result, although one must use more general curves than circles (which complicates the argument in (2).)

<u>Runge's Thm</u>. Let $K \subset\subset \mathbb{C}$ and $E \subseteq \mathbb{C}_\infty \setminus K$ s.t. E meets every component of $\mathbb{C}_\infty \setminus K$. If f is analytic in G, $K \subset\subset G$, then $\forall \varepsilon > 0$ $\exists$ rational function $R(z)$ w/ poles only in E s.t.

$$\sup_K |f - R| < \varepsilon.$$